LECTURES on RENORMALISATION and CRITICAL PHENOMENA

Based on
A Lecture Series by

TaeHwan OH

Email: Hadron@khu.ac.kr
Webpage: http://hep-thoh.khu.ac.kr

In association with

Daniel Han Kim, Sejin Kim, Sejin Lim, Kwanwoo Kim, Jeongyoung Kim

KYUNG HEE UNIVERSITY

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1 Introduction

In 1865, Maxwell discovered four equations which describes the dynamics of electromagnetic fields $\vec{E}$ and $\vec{B}$, called Maxwell's equations:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0}, \qquad & \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}, \\ \vec{\nabla}\cdot\vec{B} &= 0, \qquad & \vec{\nabla}\times\vec{B} &= \mu_0\left(\vec{J}+\epsilon\frac{\partial\vec{B}}{\partial t}\right), \end{aligned} \tag{1.1}$$

where ϵ_0 is the electric permittivity in vacuum, μ_0 the magnetic permiability in vacuum,ρ a charge density, and $\vec{J}$ a current. Then the corresponding action of the Maxwell theory is

$$S = -\frac{1}{4e^2}\int d^4x(\vec{E}^2 - \vec{B}^2). \tag{1.2}$$

In equation (1.2), we denote e^2 which is usually omitted, to emphasize the charge. With proper field redefinition, we can rewrite this equation with the fine structure constant $\alpha = e^2/4\pi\epsilon_0 \approx 1/137$ (where $\hbar = c = 1.$):

$$S = -\frac{1}{\alpha}\int d^4x(\vec{E}^2 - \vec{B}^2). \tag{1.3}$$

The action (1.3) is the classical action. However, the fundamental theory which describes the nature is a quantum theory, and the classical thoery is a macroscopic limit of corresponding quantum theory. So we can consider the *"quantization"* of action (1.3). Here, "" implies that we abused the term quantizaiton; the following description is not actually a quantization, but it is description we use in quantum field theory(QFT).

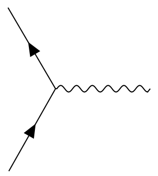
$$S = -\frac{1}{4e^2}\int d^4x\left[(F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu) + (\text{matter})\right]. \tag{1.4}$$

The first term implies the dynamics of photon, the second term $A_\mu J^\mu$ is the interaction term between photon and charge, and matter term is about the dynamics of charge. For example, if we consider an electron which is a fermion, the action is written as

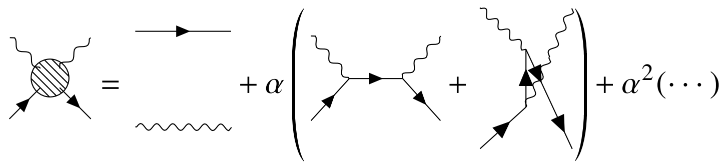
$$S = \int d^4\left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu + i\gamma^\mu A_\mu - m)\psi\right]. \tag{1.5}$$

The second term of the action (1.5) gives equation of motion $i\gamma^\mu(\partial_\mu + A_\mu)\psi = m\psi$, which is known as **Dirac equation**.

We now have the theory and we need to check wheter it suite with the experiemnts from the accelerator. To do so, we first have to calculate scattering amplitudes with the famous tool called **Feynman diagram**:

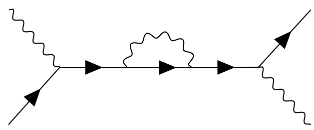
$F_{\mu\nu}F^{\mu\nu}$		$= \langle A_\mu(x)A_\nu(y)\rangle$	Photon correlation function	
$\bar{\psi}(i\gamma^\mu\partial_\mu)\psi$		$= \langle\bar{\psi}\psi\rangle$	Fermion correlation function	(1.6)
$\bar{\psi}(i\gamma^\mu A_\mu)\psi$		$= \langle A\bar{\psi}\psi\rangle$	Vertex.	

Let us consider the Compton scattering as an example. The Compton scattering is a scattering betweeen an electron and photon. The corresponding Feynman diagrams are



(1.7)

where α is the fine structure constant, that is, a coupling constant. The first term of the right hand side is a diagram that occurs no scattering. The second term has scattering however, it is just a classical effect and called **tree-level diagram**. All the quantum effects are in α^2 term or higher, and they contains **loop diagram** which can be drawn as



(1.8)

The problem is that this kind of diagrams diverge, which is not physical at all. For this diagram, you will meet integral in order of $\int d^4k\frac{1}{k^4}$ which gives $\log k$ and diverges as $k \to \infty$. However, in this lecture, we will omit all the detailed calculation and take the result that the loop diagrams which give quantum corrections diverge.

This problem was solved by very powerful method called **Renormalisation** which is developed by Kramers, Bethe, Schwinger, Feynman, Tomonaga and Dyson in 1947 1949. The key idea of renormalisation is that the infinity we have from the loop diagrams are came from the hidden infinities that the theory has. So the renormalisation is basically extracting infinities from our theory and subtract them by *renormalising* all coupling constants and fields in original(bare) Lagrangian. For example, let us consider the action (1.5). Let us denote the bare coupling constants and fields with subscript(or superscript) B, and renormalised coupling constants and fields are denoted with subscript(or superscript) R.

$$\psi_B = z_2^{1/2}\psi_R \qquad A_\mu^B = z_3^{1/2}A_\mu^R \qquad m_B = z_m m_R \qquad e_B = z_e e_R \tag{1.9}$$

In fact, $z_2 = z_3$ and from the relation between e_B and e_R, we can find the relation between α_B and α_R. All z_i could be written

$$z_i = 1 + \delta_i \tag{1.10}$$

where δ_i called **counterterm**. With this expansion, for example,

$$\begin{aligned}\bar{\psi}_B m_B \psi_B &= (1 + \frac{1}{2}\delta_2)(1 + \delta_m)(1 + \frac{1}{2}\delta_2)\bar{\psi}_R m_R \psi_R \\ &\approx \bar{\psi}_R m_R \psi_R + (\delta_2 + \delta_m)\bar{\psi}_R m_R \psi_R.\end{aligned}$$

Then we have infinities came from the first term, but those infinities are killed by the infinities from the counter terms. In this context, the physical quantities that we observed are just substraction betweeen two infinities.

This does not look so convincing, that is, it doesn't look natural but highly artificial. In fact, there is a crucial note on renormalisation. It is natural to introduce two energy scales: cut-off Λ and renormalisation sacle μ. As the consequence of renormalisation, all parameters depend on the scale μ, that is, the coupling constants are not actulally the constant:

$$e^2_{\text{eff}}(-p^2) = e^2_{\text{eff}}(\mu)\left[1 + \frac{e^2_{\text{eff}}(\mu)}{12\pi^2}\log\frac{-p^2}{\mu^2}\right] \tag{1.11}$$

The physical intuition of renormalisation was developed by Kenneth Wilson in 1971. He found that we can understand renormalisation in the context of coarse-graining.

2 Ising Model in Various Dimensions

Ising model is a model introduced by Ernst Ising in 1924, describing ferromagnetism by spin lattice system. The Hamiltonian of Ising model is written as

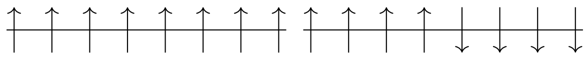
$$H(s_i;\{k_i\}) = -\frac{1}{\beta}\sum_{\{n\}} k^{(n)}\sigma^{(n)}_{i^{(n)}}, \tag{2.1}$$

where σ is called local operator and k as a coupling constant. Ising model consider the interaction between nearest neighbors, so we only consider $n = 1$ and $n = 2$ terms. However, the other terms will be regenerated after renormalisation. The terms of $n = 1$ and $n = 2$ are

$$\begin{aligned} n = 1 \quad &: \quad k^{(1)}\sigma^{(1)}_i = hs_i \\ n = 2 \quad &" \quad k^{(2)}\sigma^{(2)}_i = Js_i s_{i+1}, \end{aligned} \tag{2.2}$$

where h is Zeeman coupling and J is exchange interaction coupling constant. For $J > 0$, the system is ferromagnetic and for $J < 0$, antiferromagnetic. The Hamitonian (2.1) is then written as

$$H(s_i; J, h) = -\frac{h}{\beta}\sum_{i=1}^{N} s_i - \frac{J}{\beta}\sum_{i=1}^{N-1} s_i s_{i+1}. \tag{2.3}$$



(a) Ground state (b) First excited state

Figure 1: 1D Ising Model

The corresponding partition function of (2.3) is

$$Z(J,h,T) = \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} e^{-\beta H(s_i;J,h)} = e^{-\beta F(J,h,T)}, \tag{2.4}$$

where $F(J,h,T)$ is free energy.

2.1 1D Ising Model

Let us start with 1D case first. The most efficient way to describe this model is defining magnetization:

$$m = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle. \tag{2.5}$$

For $h \approx 0$, the ground state is the state that all the spins are aligned in the same direction, and the first excited state is the case that the spins are flipped at one point called domain wall (Figure 1). The ground state has energy $E_0 = -NJ$, entropy $S_0 = -k\log\Omega = 0$ and $F_0 = E_0 - TS_0 = -NJ$. For the first excited state, the energy is $E_1 = -(N-1)J + J = -J(N-2)$, entropy $S_1 = k_B \log N$, magnetization $m = 0$ and free energy $F_1 = -J(N-2) - k_B T\log N$. The difference of the free energy is then

$$\Delta F = F_1 - F_0 = 2J - k_B T\log N. \tag{2.6}$$

If we take a thermodynamic limit, that is, $N \to \infty$,

$$\Delta F \approxeq -k_B T \log N < 0 \tag{2.7}$$

which implies that there is no phase transition at finite temperature in 1D Ising model.

2.2 2D Ising Model

Next we consider the 2D Ising model. The 2D case is basically the same as 1D Ising model, but now we should consider 2 dimensional lattice. The (figure 2) is denoting the first excited state of 2D Ising model, where the dotted line denotes the domain wall. In this case, the difference of energy, entropy and free energy between the ground state and

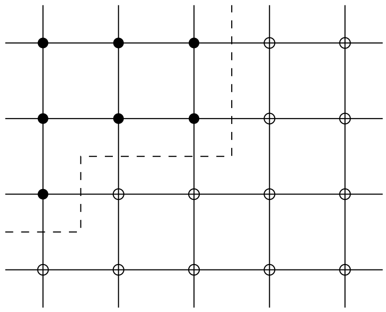


Figure 2: The first excited state of 2D Ising model

the first excited state is that

$$\begin{aligned} &\Delta E = 2Jn \\ &\Delta S \approxeq k_B \log 3^n \qquad \text{In general,} k_B \log(z-1)^n \\ \therefore \quad &\Delta F = 2Jn - k_B T \log 3^n = n[2J - k_B T \log 3] \end{aligned} \tag{2.8}$$

where n is the number of misaligned bond and z a coordination number. Roughly, if we take the thermodynamic limit, that is, $N \to \infty$, n also goes to ∞. Therefore, we can safely approximate $n = N$ in this limit. Then,

$$\Delta F \approxeq N[2J - k_B T \log 3] > 0 \tag{2.9}$$

when $T < \frac{2J}{k_B \log 3} \equiv T_c$: critical temperature. If $T > T_c$, then the magnetization $m = 0$, but for $T < T_c$, there can be $m \neq 0$. In short, there exists the long range ordering, or **phase transition**, for $d > 1$ and we call $d_L = 1$ as the lower critical dimension.

2.3 Revisit 1D Ising Model

For simplicity, as we did in section 2.1, we will take $h = 0$. We also apply no periodic boundary condition, that is,

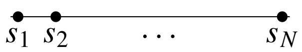


Figure 3: 1D Ising model without periodic boundary condition

Then the Hamiltonian and the partition functions are

$$\begin{aligned} H &= -J \sum_{i=1}^{N-1} s_i s_{i+1} \\ Z &= \sum_{s_1} \cdots \sum_{s_N} e^{J \sum s_i s_{i+1}}. \end{aligned} \tag{2.10}$$

If we define $\eta_i \equiv s_i s_{i+1}$ $(i = 1 \cdots N-1)$, we can rewrite the partition function,

$$\begin{aligned} Z &= \sum_{\eta_1} \cdots \sum_{\eta_{N-1}} \sum_{s_N} e^{J \sum \eta_i} \\ &= \sum_{s_N = \pm 1} \left[\sum_{\eta = \pm 1} e^{J\eta} \right]^{N-1} \\ &= \sum_{s_N = \pm 1} (e^J + e^{-J})^{N-1} = 2^N \cosh^{N-1} J \end{aligned} \tag{2.11}$$

If we redefine J as $J = \beta \bar{J}$, the partition function in (2.11) becomes

$$Z = 2^N \cosh^{N-1} \bar{J}\beta. \tag{2.12}$$

We can take the thermodynamic limit ($N \to \infty$) to (2.11), then the partition function and the free energy are written as

$$\begin{aligned} Z[J] &= 2^N \cosh^N J = e^{JN}(1 + e^{-2J})^N \\ F &= -k_B T \log Z = -N k_B T (J + \log(1 + e^{-2J})). \end{aligned} \tag{2.13}$$

If we define a quantity f a free energy density as

$$f \equiv \frac{F}{N} = \begin{cases} -J/\beta & T \to 0 \\ -k_B T \log 2 & T \to \infty. \end{cases} \tag{2.14}$$

In case of the first case, $J \to \infty$ when $T \to 0$ since $J = \bar{J}\beta$, and this makes the log term zero. For the second case, $J \to 0$ when $T \to \infty$, and this gives $\log 2$ term.

We can calculate other thermodynamical variables with (2.11). For example, the internal energy U is

$$\begin{aligned} U = \langle E \rangle &= -\frac{\partial}{\partial \beta} \log Z \\ &= -N k_B T J \tanh J \end{aligned} \tag{2.15}$$

and the specific heat C is

$$C = \frac{dU}{dT} = -\frac{1}{k_B T^2} \frac{\partial U}{\partial \beta} = \frac{NJ}{k_B} \operatorname{sech}^2 J. \tag{2.16}$$

Another important quantity that we can calculate is the correlation function $G(i, j)$, which is defined as

$$G(i, j) = \langle (s_i - \langle s_i \rangle)(s_j \langle s_j \rangle) \rangle = \langle s_i s_j \rangle, \tag{2.17}$$

this is because the magnetization is zero at finite temperature. The correlation function

$G(i, i+1)$ is calculated as following:

$$\begin{aligned} G(i,i+1) &= \frac{1}{Z}\sum_{\{s_i\}} s_i s_{i+1} e^{J_1 s_1 s_2 + \cdots + J_{N-1}s_{N-1}s_N}\Bigg|_{J_1=\cdots J_{N-1}=J} \\ &= \frac{1}{Z}\frac{\partial}{\partial J_i}\left[\sum_{\{s_i\}} e^{J_1 s_1 s_2 + \cdots + J_{N-1}s_{N-1}s_N}\right]\Bigg|_{J_1=\cdots=J_{N-1}=J} \\ &= \frac{\partial}{\partial J_i}\log Z[\{J_i\}]\Bigg|_{J_i=J} = \tanh J. \end{aligned} \tag{2.18}$$

Likewise, we can calculate $G(i, i+2)$:

$$G(i,i+2) = \frac{1}{Z}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_{i+1}}\left[\sum_{\{s_i\}} e^{J_1 s_1 s_2 + \cdots + J_{N-1}s_{N-1}s_N}\right]\Bigg|_{J_i=J} = \tanh^2 J. \tag{2.19}$$

If we do the same calculation iteratively, it can be easily shown that for arbitrary positive integer j,

$$G(i, i+j) = \tanh^j J. \tag{2.20}$$

Now, let us consider the long range ordering, that is, phase transition. If there exist such phase transition, then $G(i, j) = 1$ for all j. Let us define a correlation lenght ξ such that

$$G(i,i+j) = e^{-j/\xi} \qquad \xi = \frac{1}{\log\coth J}. \tag{2.21}$$

Physically, ξ is the approximate length for sustaining j^{th} spin information.

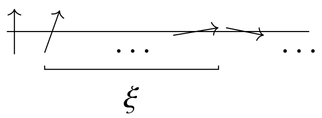


Figure 4: correlation length

By (2.21), the long range ordering occurs when $\xi \to \infty$. This is equivalent with $\log\coth J = 0$, that is $\coth J = 1$. Since $\coth\infty = 1$, this occurs when $\beta \to \infty$ or $T \to 0$. Thus, for finite T, correlation length does not diverge. This coincide well with our previous result: there is no phase transition in 1D Ising model.