Exercises for Lectures of Renormalisation and Critical Phenomena

Exercises for Lecture 1

We will discuss about symmetry property for 1d Ising model. Since the spin has the character of the angular momentum, it is a pseudo-vector under space-inversion or time-reversal transformation $\mathbb{T}$.

Exercise 1-1

Show that angular momentum vector $\vec{J}$ is antisymmetric under time-reversal transformation, that is $\mathbb{T}\vec{J}\mathbb{T}^{-1} = -\vec{J}$.

Exercise 1-2

Is Hamiltonian for 1d Ising model invariant under time-reversal transformation? What is the condition for time-reversal symmetry?

Exercise 1-3

Show that partition function and free energy have time-reversal symmetry independent from the condition in the Exercise 1-2.

Exercises for Lecture 2

We have covered Curie-Weiss mean field theory in the lecture. There is an another type of mean field theory called Bragg-Williams mean field theory.

Let us consider a hypercube spin lattice in dimension d.

Exercise 2-1

Let suppose be that N_+ is the number of the up-spins, N_- is the number of the down-spins, and N is the total number of the spins, so $N_+ + N_- = N$. Show that m, E, s are represented as (1).

$$\begin{cases} m = \dfrac{N_+ - N_-}{N} \\ E = -N\bar{J}dm^2 - N\bar{h}m \\ s = \dfrac{1}{N}S = k_B\left(\log 2 - \dfrac{1}{2}(1+m)\log(1+m) - \dfrac{1}{2}(1-m)\log(1-m)\right) \end{cases} \tag{1}$$

Exercise 2-2

Show that free energy density can be written as (2)

$$f = -\bar{J}dm^2 - \bar{h}m - k_BT\log 2 + \frac{k_BT}{2}(1+m)\log(1+m) + \frac{k_BT}{2}(1-m)\log(1-m) \tag{2}$$

and near T_c it also can be written as (3).

$$f(m) = -k_BT\log 2 - \bar{h}m + \frac{1}{2}k_B(T-T_c)m^2 + \frac{1}{12}k_BTm^4 + \cdots \tag{3}$$

Note that (3) is the first example of Ginzburg–Landau theory.

Exercise 2-3

At the minimum of $f(m)$, show that Bragg–Williams theory is identical to Curie–Weiss mean field theory. That means show that m is given by

$$m = \tanh(h + 2dJm) \tag{4}$$

(Hint: You should use the identity (5) during the calculation)

$$\frac{1}{2}\log\left(\frac{(1+m)}{(1-m)}\right) = \operatorname{arctanh}(m) \tag{5}$$

Exercises for Lecture 3

We only covered the Landau free energy density up to fourth order η^4. However, if the coefficient a_4 can becomes negative, we should consider up to η^6 term. In this system, we can find tricritical point in the phase diagram for matter system (i.e. phase diagram of water).
Let the Landau free energy density is given as (6)

$$f_L = a\eta^2 + b\eta^4 + c\eta^6 \tag{6}$$

Exercise 3-1

Show that there is no stable state when $a < 0$ except $\eta = 0$ by extremising the Landau free energy density.

Exercise 3-2

Investigate the case for $a > 0$, $b > 0$ and $a > 0$, $b < 0$. What is the difference between two cases?

Exercise 3-3

Draw the phase diagram in the ab plane. Find the tricritical point is placed on $a = 0,\ b = 0$.
Note: In the case of the real material, the corresponding order parameter is the density ρ and the external field h (I omitted here) is the chemical potential μ.

Exercises for Lecture 4

In these exercises, we will calculate the partition function for the harmonic oscillator, the only possible partition function with simple calculations. First of all, we will recall the Gaussian integral / Gaussian functional integral.

Exercise 4-1

Prove these Gaussian integral formulation

$$\int dx\ e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}} \tag{7}$$

and

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\vec{\eta}^T M\vec{\eta}+\vec{A}^T\vec{\eta}} = \sqrt{\frac{(2\pi)^N}{\det M}}e^{\frac{1}{2}\vec{A}^T M^{-1}\vec{A}} \tag{8}$$

where N is the dimension of the vector $\vec{\eta}$ and $\vec{A}$.
(Hint: To prove (8), it is better to choose the basis to the eigenvectors for matrix M, then (8) becomes just a multiple integration of (7).)

Exercise 4-2

In the classical mechanics, we had shown that the Hamiltonian for the 1d harmonic oscillator is given as (9)

$$H(q,p) = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 \tag{9}$$

Here, we set $m = 1$ for simplicity. Then, show that the partition function for a single harmonic oscillator is written as (10)

$$\mathcal{Z}_1 = \frac{k_B T}{\hbar\omega} \tag{10}$$

So, the (canonical) partition function will be given $Z = Z_1^N$ where N is the number of the oscillators.

Exercise 4-3

In quantum mechanics, the Hamiltonian for the single harmonic oscillator is written as (11)

$$H_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{11}$$

Show that the partition function will be given as (12)

$$\mathcal{Z} = e^{-\frac{1}{2}\frac{\hbar\omega}{k_B T}}\frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}} = \frac{1}{2\sinh(\hbar\omega\beta/2)} \tag{12}$$

As we discussed in the lecture, the partition function is related to the functional integral in this way

$$Z = \int_{PB} \mathcal{D}x \; e^{-\beta H} \tag{13}$$

where PB means that the periodic boundary condition for x. From this formulation, one can find the partition function in this way

$$\mathcal{Z} = \int \mathcal{D}x e^{-\int_0^\beta d\tau\left(\frac{1}{2}x'^2+\frac{1}{2}\omega^2 x^2\right)} \tag{14}$$

Exercise 4-4

From the periodic boundary condition we can expand x into the Fourier series

$$x(\tau) = \frac{x_0}{\sqrt{\beta}} + \frac{1}{\sqrt{\beta}}\sum_{n\neq 0} x_n e^{2\pi i n\tau/\beta} \tag{15}$$

Then, show that the (14) can be expressed

$$\int \frac{dx_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \frac{dx_n dx_{-n}}{2\pi} e^{-\frac{1}{2}\omega^2 x_0^2 - \frac{1}{2}\sum_{n\neq 0}\left(\left(\frac{2\pi i n}{\beta}\right)^2+\omega^2\right)x_n x_{-n}} \tag{16}$$

and finally we can find

$$\mathcal{Z} = \frac{1}{\omega}\prod_{n=1}^{\infty}\frac{1}{\left(\frac{2\pi n}{\beta}\right)^2+\omega^2} \tag{17}$$

The equation (17) looks far different with (12). But if we treat (17) mathematically and carefully, we can find that (17) is exactly identical to (12). This method is called the regularisation.

Exercises for Lecture 5

In this exercise, we will discuss about Kadanoff's block spin transformation to 1D Ising model. We will work it with a very similar strategy to 2D Ising model in the lecture. Let us consider a N spin-lattice system.

Exercise 5-1

Write a Hamiltonian including only interaction between nearest neighbour, so there is no Zeeman term in the Hamiltonian. Also write corresponding partition function.

Exercise 5-2

Now, we will sum-over the spins on the odd sites. For a while, we focus on the second site. As we followed in the lecture, effective Hamiltonian will be written as:

$$\sum_{s_2=\pm} e^{Js_2(s_1+s_3)} = e^{J's_1s_3+F} \tag{18}$$

Then show that J' is written as

$$\tanh J' = \tanh^2 J \tag{19}$$

Exercise 5-3

From equation (19), argue that

$$G(i, i+n; J) = G(i, i+n/2; J') \tag{20}$$

where $G(i,j;J)$ is the correlation function between two sites i and j with coupling constant J. Also show that after decimation, the correlation length become half of the original length that means $\xi(K') = 1/2\xi(K)$

Exercises for Lecture 6

In this exercise, we will calculate Green's function (or two-point correlation function) for Gaussian model in arbitrary dimension. To do this, first we need to recall the definition of the Green's function.

Exercise 6-1

Recall the Fourier transform of a given function

$$f(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k) e^{ik\cdot x} \tag{21}$$

From the definition of the Green's function

$$(-\nabla^2 + r)G(x, y) = \delta^d(x - y) \tag{22}$$

show that the Green's function can be expressed as (23)

$$G(x - y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{k^2 + r} \tag{23}$$

Exercise 6-4

Now we will restrict our dimensionality into four to correspond to our space. So we will evaluate (23) by setting $d = 3$. After imposing rotational symmetry in the momentum space, (23) becomes into more familiar form.

$$G(x - y) = \frac{1}{(2\pi)^2} \int_0^\infty d|k||k|^2 \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \frac{1}{k^2 + r} e^{i|k||x-y|\cos\theta} \tag{24}$$

Then we can find the Yukawa potential with parameter r after evaluate (24)

$$G(x - y) = \frac{1}{|x - y|} e^{-r|x-y|} \tag{25}$$

Exercise 6-5

As we discussed in the lecture, parameter r plays same role with mass. Argue that the correlation length of the system is related to the mass of the field by (25).

Exercises for Lecture 9

In the lecture, we have discussed about basic group theory. We will check about basic group theory from the familiar case SO(3) and SU(2) which we already dealt in the classical mechanics and quantum mechanics courses.

Exercise 9-1

First, we are starting from the Euler angle. As we have studied in the classical mechanics, any rotation in the 3d space can be written in the combination of three angles $\alpha \in [0, 2\pi), \beta \in [0, \pi],$ and $\gamma \in [0, 2\pi)$ such that

$$g(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

Show that $g(\alpha, \beta, \gamma)$ is really an element of SO(3) group.

Exercise 9-2

Now, we will move on another system. Let us consider a 2-sphere with radius 1 without north pole $N = (0, 0, 1)$.

$$P \in S^2 = \left\{(x, y, z) | x^2 + y^2 + z^2 = 1\right\} / (0, 0, 1)$$

Then we can parametrise all point by stereographic projection. Let us consider a line between N and a point on the sphere $P = (x, y, z)$ can be parametrised by $t \in \mathbb{R}$ in this way.

$$f(t) = N + t(N - P) \tag{27}$$

Then show that P can be rewritten in a point in the xy plain (or for later convenience it is a point in the complex plain)

$$(x, y, z) \mapsto (p, q) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \equiv \xi = p + iq \tag{28}$$

Exercise 9-3

Now, we will argue that ξ will be transformed by Möbius transformation under SO(3) rotation for (x, y, z) More precisely, ξ is transformed like $SL(2, \mathbb{C})$. Möbius transformation is defined as below

$$\xi \mapsto \xi' = \frac{a\xi + b}{c\xi + d}, \quad a, b, c, d \in \mathbb{C} \tag{29}$$

Show that under rotation along x, y and z axis, respectively, and then ξ follows the Möbius transformation with $ad - bc = 1$. This statement is obviously equivalent with that the corresponding matrix

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{30}$$

has unit determinant.

Note that this SL$(2, \mathbb{C})$ is not unique. If (a, b, c, d) is an element of the SL$(2, \mathbb{C})$, then $(-a, -b, -c, -d)$ is also the element of the SL$(2, \mathbb{C})$. So, we can conclude that one SO(3) element is corresponding to two SL$(2, \mathbb{C})$ element, namely, $g \mapsto \pm m$ where $g \in$ SO(3) and $m \in$ SL$(2, \mathbb{C})$. It is said that SL$(2, \mathbb{C}) \simeq$ SO$(3) \times$ SO(3) or SL$(2, \mathbb{C})$ is the universal covering group of SO(3).

Exercise 9-4

By using the Euler angle, show that corresponding SL$(2, \mathbb{C})$ can be written as

$$h[g(\alpha, \beta, \gamma)] = \pm \begin{pmatrix} \cos\frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}} & i\sin\frac{\beta}{2} e^{\frac{\alpha-\gamma}{2}} \\ i\sin\frac{\beta}{2} e^{-i\frac{\alpha-\gamma}{2}} & \cos\frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}} \end{pmatrix} \tag{31}$$

and also $h[g(\alpha, \beta, \gamma)]$ is an element of SU(2) group. [Hint: First prove that h is preserving the multiplication between g]

Exercise 9-5

Finally, we will analyse the SU(2) group. By definition, we can represent the SU(2) group as a 2×2 matrix with unitarity and unit determinant. For an element A of SU(2) group,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{32}$$

show that this matrix is parameterised by unit 3-sphere S^3 and also it can be represented by linear combination of 2×2 identity and Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{33}$$

In terms of the algebra, we can easily check that $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$

When we remind SO(3) representation in quantum mechanics, we can easily guess that Pauli matrices are coincidental with spin-$\frac{1}{2}$ representation. It is a key to understand Fermion in quantum mechanics (e.g. spin precession).