KYUNG HEE
UNIVERSITY

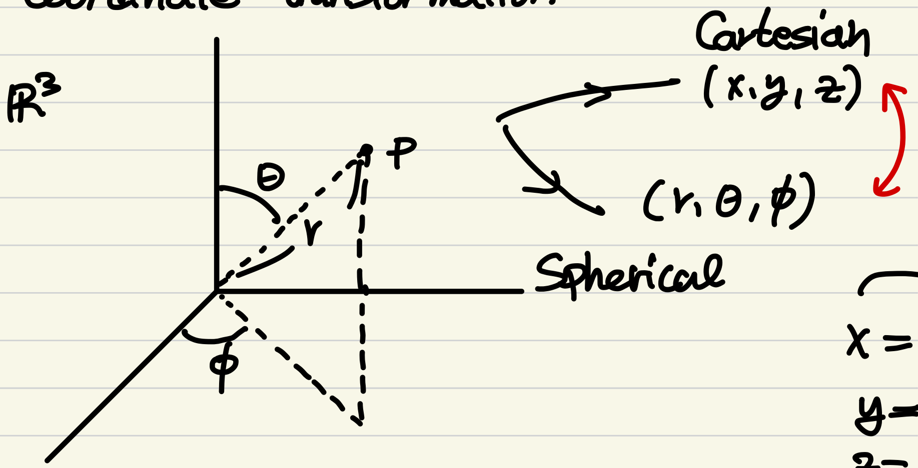
The World is
a Global Village and
the Peoples of the World
are One Human Family.
May We Strive for
Peace and Humanity
with the Spirit of
Global Cooperation
Society.

Lectures on the Classical Field Theory

I. Preliminaries

I.1. Multilinear algebra

- Coordinate transformation



???

$$\Downarrow$$

$$\overbrace{\begin{aligned} x&=r\sin\theta\cos\phi\\ y&=r\sin\theta\sin\phi\\ z&=r\cos\theta \end{aligned}}$$

$$\Downarrow$$

$$\begin{aligned} x&=f(r,\theta,\phi)\\ y&=g(r,\theta,\phi)\\ z&=h(r,\theta,\phi) \end{aligned}$$

In general, original coordinate system $\{q_i = q_1,\cdots,q_n\}$

changed coordinate system $\{q'_i = q'_1,\cdots,q'_n\}$

$$q_i = f_i(q'_1,\cdots,q'_n)$$: coordinate function

$$q'_j = \sum_{i=1}^{n}\frac{\partial q'_j}{\partial q_i}q_i \underset{\substack{\text{Einstein's}\\ \text{convention}}}{=} \frac{\partial q'_j}{\partial q_i}q_i$$

Repeated Indices ⇒ summed index

$$= A^i{}_j\, q_i$$: covariant transformation

$$\frac{\partial}{\partial q_i} = \frac{\partial q'_j}{\partial q_i}\frac{\partial}{\partial q'_j} = A^i{}_j \boxed{\frac{\partial}{\partial q'_j}} \rightsquigarrow \text{basis of vector } e'_j$$

$$dq_i = \frac{\partial q_i}{\partial q'_j}\,dq_j = \underline{(A^{-1})_i{}^j}\,\boxed{dq_j} \rightsquigarrow \text{basis of co-vector (dual) } \bar{e}'_j$$

↪ form

$$\Downarrow$$

Contravariant transformation

$$\bar{e}'_j \cdot e^i = \delta^i_j$$

$$\vec{A}\cdot\vec{B} = (A^j\bar{e}_j)\cdot(B_i e^i) = A^j B_i(\bar{e}_j\cdot e^i) = A^i B_i$$

$\bar{e}_j = \langle j|$ $\quad e^i = |i\rangle$

$A|i\rangle \leftrightsquigarrow \langle j|A^{-1}$ $\quad$ |⇒ Under the transformation by A, the norm is not changed.

"Isometry" ⇝ group G

ex) $\mathbb{R}^3 \rightsquigarrow G=SO(3)$

$$e^{i} \to e'^{\bar{i}} \qquad \bar{e}_{j} \to \bar{e}_{\bar{j}}$$

- Tensor
  - Vector (representation of G)

$$V_{\bar{i}} \longrightarrow A^{\bar{j}}_{\ \bar{i}} V_{\bar{j}}$$

  - Covector

$$W^{\bar{i}} \longrightarrow (A^{-1})_{\bar{j}}^{\ \bar{i}} W^{\bar{j}}$$

※ For $\mathbb{R}^3$, there is no "difference" between vector and co-vector due to the "$\bar{e} \cdot e = \delta$".

  - (p,q)−tensor (p+q)-rank tensor

$$T_{\bar{i}_1 \cdots \bar{i}_p}^{\ \ \bar{j}_1 \cdots \bar{j}_q} \longrightarrow (A)^{k_1}_{\ \bar{i}_1} \cdots (A)^{k_p}_{\ i_p} (A^{-1})_{l_1}^{\ \ j_1} \cdots (A^{-1})_{l_q}^{\ \ \bar{j}_q} T_{\cdots}^{\ \cdots}$$

* (Anti−) Symmetric tensor

$$T_{\bar{i}_1 \cdots \bar{i}_p \cdots \bar{i}_q \cdots i_n} = \underset{\text{anti-symmetric}}{\overset{\text{symmetric}}{\pm}} T_{i_1 \cdots \underset{p\text{th}}{\bar{i}_q} \cdots \underset{q\text{th}}{\bar{i}_p} \cdots i_n}$$

ex) Matrix M

$$M_{i\bar{j}} = \pm M_{j\bar{i}}$$

cf) We call an anti-symmetric (0,q)−tensor (rank q contravariant ten) as q−form

$$\omega^{\bar{i}_1 \cdots \bar{i}_q} = -\omega^{\bar{i}_1 \cdots i_b \cdots i_a \cdots \bar{i}_q}$$

I.2. Special Theory of Relativity

- Principle of Relativity (1905, Einstein)

"If a physics law hold in a coordinate system, it is also held to any other systems moving in uniform translation relatively to the reference one.

frame — moving with constant velocity

Maxwell theory
Electrodynamics

⇒ "c" speed of light
↳ Lorentz transformation $\Lambda^{\mu}_{\ \nu}$

⇒ Space−time four−vector

$\mu, \nu = 0, \underbrace{1, 2, 3}$
0: time, $1,2,3$: space $\bar{i}$

⇓

Forming group SO(1,3)

- Minkowski metric

SO(3) $\quad M^T \delta M = \mathbb{1}$

SO(1,3) $\quad M^T \underbrace{\begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{\text{Minkowski metric } \eta_{\mu\nu}} M = \mathbb{1}$

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

* Inverse metric $\eta^{\mu\nu}$ $\quad \eta^{\mu\rho}\eta_{\rho\nu} = \delta^{\mu}_{\nu}$

* Symmetric $\eta_{\mu\nu} = \eta_{\nu\mu}$

* Norm $\quad V\cdot W = V^{\mu}W^{\nu}\eta_{\mu\nu} = V^0W^0 - V^1W^1 - V^2W^2 - V^3W^3$

* One-to-one map between vector ⟷ covector

$$V_{\mu} = \eta_{\mu\nu}V^{\nu} \qquad W^{\mu} = \eta^{\mu\nu}W_{\nu}$$

(i.e. $V_0 = V^0$, $V_i = -V^i$, $\cdots$)

$$T_{\mu\nu} = T_{\mu}{}^{\rho}\eta_{\rho\nu}$$

- Four-vector

$$X^{\mu} = (x^0, x^1, x^2, x^3)$$

$$X^2 = X^{\mu}X_{\mu} = X^{\mu}X^{\nu}\eta_{\mu\nu} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

ex) position $(ct, x, y, z) = x^{\mu}$

$$x^2 = c^2t^2 - x^2 - y^2 - z^2$$

energy-momentum $(cE, P_x, P_y, P_z) = p^{\mu}$

$$c^2E^2 - (P_x)^2 - (P_y)^2 - (P_z)^2 = m^2c^4$$

Under Lorentz transformati

$$\left[\begin{aligned} V'^{\mu} &= \Lambda^{\mu}{}_{\nu}V^{\nu} \\ W'_{\mu} &= (\Lambda^{-1})_{\mu}{}^{\nu}W_{\nu} \end{aligned}\right. \qquad \eta_{\mu\nu} = (\Lambda^{-1})_{\mu}{}^{\rho}(\Lambda^{-1})_{\nu}{}^{\sigma}\eta_{\rho\sigma}$$

$$\vdots$$

II. Klein-Gordon Equation

II.1. Klein-Gordon Equation

Special relativity (1905)

Schrödinger equation (1925) ⟹ Klein-Gordon equation (1926) ⇓ Dirac equation (1928)

$$-i\frac{\partial}{\partial t}\psi = H\psi$$

· First trial

$$-i\frac{\partial}{\partial t}\psi = \overset{\to E}{H}\psi = \pm\sqrt{-\vec{\nabla}^2 + m^2}\ \psi$$

two weaknesses

① negative sign

② Non-locality

⇒ square both side ⇝ Klein-Gordon equation

$$-\frac{\partial^2}{\partial t^2}\psi = -\vec{\nabla}^2 + m^2\psi$$

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\psi + m^2\psi = \underbrace{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)}\psi + m^2\psi = 0$$

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \eta_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$$

: d'Alembertian

$$\boxed{(\Box + m^2)\phi = 0}$$; Klein-Gordon equation

m^2 → mass of field ϕ

- Fourier decomposition

$$\phi(x) = \int_{\mathbb{R}^3}\frac{d^3\vec{k}}{(2\pi)^{3/2}}\varphi_{\vec{k}}(t)\,e^{-i\vec{k}\cdot\vec{x}}$$

Reality condition : $\phi^*(x) = \phi(x)$

$$\phi^*(x) = \int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\varphi^*_{\vec{k}}(t)\,e^{+i\vec{k}\cdot\vec{x}} = \phi(x)$$

⇒ $\varphi_{\vec{k}}(t) = \varphi_{-\vec{k}}(t)$: Reality condition

To manifest Lorentz covariance

$$\varphi_{\vec{k}} = \frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{+i\omega_{\vec{k}}t} + b_{\vec{k}}e^{-i\omega_{\vec{k}}t}\right) \qquad \omega_{\vec{k}}^2 = k^2 + m^2$$

Reality condition ⇒ $b_{-\vec{k}} = a^*_{\vec{k}}$

$$\Rightarrow \phi(x) = \int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\Big(\underbrace{a_{\vec{k}}e^{ik\cdot x}}_{\text{negative frequency}} + \underbrace{a^*_{\vec{k}}e^{-ik\cdot x}}_{\text{positive frequency}}\Big)$$

quantization → anti-particle/annihilation

→ particle/creation

- Solving Klein-Gordon

$$(\Box + m^2)\,\phi(x) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\varphi_{\vec{k}}(t) + (+\vec{k}^2 + m^2)\,\varphi_{\vec{k}}(t) = 0$$

$\Rightarrow$ Harmonic oscillator

Note
- We can deal with a (classical) field as a collection of infinite harmonic oscillators.
- After quantization, a and a^* are promoted to the harmonic oscillator creation / annihilation operators a and $a^\dagger$.

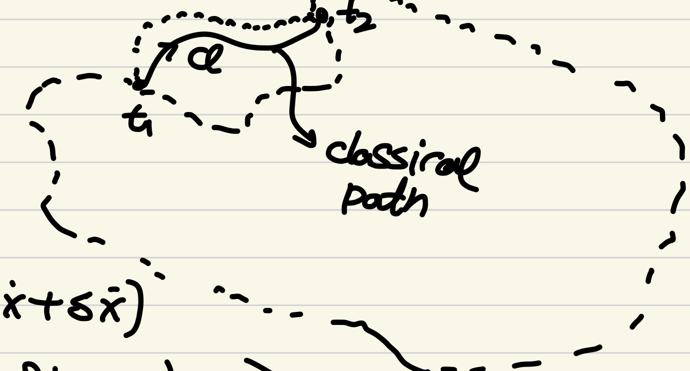
II.2. Scalar Lagrangian

- Recall : Lagrangian mechanics

Hamilton's principle $\Rightarrow$ Classical path = extremising action functional

$$S[x] = \int_{t_1}^{t_2} dt\, L(x, \dot{x})$$

$$\delta S \overset{!}{=} 0$$



$$\Rightarrow S[x+\delta x] - S[x] = 0$$

$$S[x+\delta x] = \int_{t_1}^{t_2} dt\, L(x+\delta x, \dot{x}+\delta\dot{x})$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}\right) + \cancel{\mathcal{O}(\delta^2)} + S[x]$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x + \cancel{\left(\frac{\partial L}{\partial \dot{x}}\right)\Big|_{t_1}^{t_2}} + S[x]$$

$$\delta S = 0,\ \forall\, \delta x \Longrightarrow \boxed{\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0} \quad \text{Euler-Lagrange equation}$$

$$\frac{\partial L}{\partial \dot{x}^i} = p_i \quad : \text{canonical momenta}$$

- Field equation from the Lagrangian

$$S[\phi] = \underbrace{\int dt\, L(\phi, \partial\phi)}_{\text{not Lorentz covariant form}} \qquad \left[\partial_\mu = \frac{\partial}{\partial x^\mu}\right]$$

$$= \int d^4x\, \mathcal{L}(\phi, \partial\phi) \qquad L(\phi, \partial\phi) = \int d^3\vec{x}\, \overbrace{\mathcal{L}(\phi, \partial\phi)}^{\text{Lagrangian density}}$$

$$\delta S = 0 \Rightarrow S[\phi + \delta\phi] - S[\phi] = 0$$

$$= \int d^4x \left[\mathcal{L}(\phi+\delta\phi, \partial\phi + \delta(\partial\phi)) - \mathcal{L}(\phi, \partial\phi)\right]$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \cancel{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]_{bdy}}$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi = 0, \ \forall \delta\phi$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0}$$ Euler-Lagrange equation

- Scalar Lagrangian

$$\mathcal{L}(\phi, \partial\phi) = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2}\phi(\Box + m^2)\phi \quad + (\text{boundary})$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (m^2 + \Box)\phi = 0$$ ⇒ Klein-Gordon equation

II.3. Complex Scalar Theory

ϕ_1, ϕ_2 : Real scalar field

$\phi = \phi_1 + i\phi_2$ ⇒ Complex scalar field

Condition : Lagrangian should be real and Lorentz invariant

$$\mathcal{L}(\phi, \phi^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) + m^2 |\phi|^2$$

$$\hookrightarrow \begin{cases} (\Box + m^2)\phi = 0 \\ (\Box + m^2)\phi^* = 0 \end{cases}$$

<u>Note</u> After quantization, ϕ can be regarded as a particle and ϕ^* can be regarded as an anti-particle.

particle annihilation / anti-particle creation — particle creation / anti-particle annihilation

$$\begin{cases} \phi = \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \frac{1}{(2\pi)^{3/2}} \left(a_{\vec{k}} e^{ik\cdot x} + a^*_{\vec{k}} e^{-ik\cdot x} \right) \\ \phi^* = \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \frac{1}{(2\pi)^{3/2}} \left(b_{\vec{k}} e^{ikx} + b^*_{\vec{k}} e^{-ik\cdot x} \right) \end{cases}$$

particle creation / anti-particle annihilation — particle annihilation / anti-particle creation

- Gauge theory : First-look

<u>Recall</u> In Maxwell theory

$$A_\mu = (\phi, \vec{A}) \longrightarrow A_\mu + \partial_\mu \alpha = \left(\phi - \frac{\partial \alpha}{\partial t}, \vec{A} + \vec{\nabla}\alpha\right)$$

$\Big[$ Maxwell equation
Electric/ Magnetic field $\Big]$ has not changed under this transf.

In QM

$|\psi\rangle \to e^{i\alpha}|\psi\rangle$

↳ $\Big[$ Schrödinger equation
Probability $\Big]$ does not change

⇒ Under this transformation, physical quantities are invariant
: Gauge transformation

$\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ ∴ Global U(1) transf.

$$\left[\begin{array}{l} \mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) + m^2|\phi|^2 \\ (\Box + m^2)\phi = 0 \end{array}\right\rangle \Rightarrow$$ Unchanged under transformation.

⇒ Complex scalar theory is a first example of gauge theory

III. Noether Theorem

III.1. The Law of Conservation

$$j^\mu = j^\mu(\phi, \partial\phi)$$: conserved current

- Conservation law

$$\partial_\mu j^\mu = 0 \quad \text{when} \quad (\Box - m^2)\phi = 0$$

Let $j^\mu = (\rho, \vec{J})$

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$: continuity equation

- Charge

$$Q_V = \int_V d^3\vec{x}\, \rho(x)$$

$$\frac{d}{dt} Q_V = \int_V d^3x \, \frac{d}{dt}\rho(x)$$

$$= -\int_V d^3x \, \vec{\nabla}\cdot\vec{J} = -\int_{\partial V} d^2\vec{S}\cdot\vec{J}$$

⇒ If Q_V is a conserved charge, then Q_V is never neither created or annihilated.

⇒ We can regard the "total charge" of the region as a constant of motion.

Noether thm.

E.g.) j^μ : electric current ⟷ $U(1)$
energy-momentum ⟷ $R^{1,3}$
angular momentum ⟷ $SO(1,3)$
⋮

III.2. Transformation of Field

- Transformation
  - Case 1 : vector rotation

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow[\text{2d rotation}]{} v' = \begin{pmatrix} a' \\ b' \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{R} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$$

$$\Rightarrow V'^i = R(\theta)^i{}_j V^j$$: $(SO(2))$ transformation

- Case 2 : Phase shift of wave function

$$\psi \longrightarrow \psi' = e^{i\theta}\psi$$

In general, $\phi \longrightarrow \phi' = f(\theta_i)\phi$

$\left[\begin{array}{ll} \text{Global} & \theta_i = \theta \\ \text{Local} & \theta_i = \theta(x) \end{array}\right.$ $\left[\begin{array}{lll} \text{Continuous} & f(\theta_i):\text{cont.} & \rightsquigarrow \text{Lie group} \\ \text{Discrete} & f(\theta_i):\text{disc.} & \rightsquigarrow \text{Discrete group} \end{array}\right.$

- Infinitesimal transformation

$$\frac{\delta \phi}{\delta \theta_i} \equiv \delta_i \phi = \frac{\partial \phi'}{\partial \theta_i}\bigg|_{\theta_i=0}$$

$$\delta_i \phi = \underbrace{\left(\frac{\partial f}{\partial \theta_i}\bigg|_{\theta_i=0}\right)}_{\equiv \alpha_i \rightsquigarrow \text{Lie algebra}} \phi \equiv \alpha_i \phi$$ · Linearised equation

ex) space-time translation

$$\phi'(x^\mu) = \phi(x^\mu + \Delta^\mu) \Rightarrow \delta_{\Delta_\mu} \phi = \partial_\mu \phi$$

ex) scaling transformation

$$\phi'(x) = e^\lambda \phi \Rightarrow \delta_\lambda \phi = \phi$$

ex) 3d rotation of vector

$$\vec{V}' = R_z(\alpha) R_y(\beta) R_x(\gamma) \vec{V}$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix} \quad R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ & 1 & \\ -\sin\beta & & \cos\beta \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & & \\ & \cos\gamma & -\sin\gamma \\ & \sin\gamma & \cos\gamma \end{pmatrix}$$

HW. Let $\delta_\alpha \vec{V} = T_z V$, $\delta_\beta \vec{V} = T_y V$, $\delta_\gamma V = T_x V$

Then what is the commutator relation between $\{T_x, T_y, T_z\}$?

III.3. The Noether's (First) Theorem

Consider $\mathcal{L} = \mathcal{L}(\phi, \partial\phi)$, Symmetry transformation $\delta_i \phi$

$$\delta_i \mathcal{L} \overset{!}{=} \partial_\nu W^\nu = \frac{\partial}{\partial \theta_i} \mathcal{L}(\phi', \partial\phi')\bigg|_{\theta_i=0}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi'} \frac{\partial \phi'}{\partial \theta_i} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi')} \frac{\partial(\partial_\mu \phi')}{\partial \theta_i}\bigg|_{\theta_i=0}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \delta_i \phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_i \phi + \partial_\mu \overbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_i \phi\right)}^{j_i^\mu}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \delta_i \phi + \partial_\mu j_i^\mu, \quad \forall\, \delta_i \phi$$

$$\Rightarrow \partial_\mu(\underbrace{j_i^\mu - W^\mu}_{J_i^\mu}) = -\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \overset{\checkmark}{=} 0$$

when on-shell (eom is held)

$$\Rightarrow \boxed{J_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_i \phi - W^\mu}$$ Noether's first theorem.

$\downarrow$ Noether current
corresponding to the symmetry (generator) θ_i

$Q_i = \int d^3\vec{x}\, J_i^0$: Noether charge

II.4. Examples of Noether current

- Translation symmetry

$$\delta_\rho \phi = \partial_\rho \phi$$

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi) + \frac{1}{2} m^2 \phi^2$$

$$\delta_\mu \mathcal{L} = -\frac{1}{2}(\partial_\nu \partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2}(\partial_\nu \phi)(\partial^\nu \partial^\mu \phi) + \frac{1}{2} m^2\left((\partial^\mu \phi)\phi + \phi \partial^\mu \phi\right)$$

$$= -\frac{1}{2}\partial^\mu\left[(\partial_\nu \phi)(\partial^\nu \phi)\right] + \frac{1}{2} m^2 \partial^\mu(\phi^2)$$

$$= \partial^\mu \eta_{\mu\rho} \mathcal{L}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\delta^\mu_\nu \partial^\nu \phi = -\partial^\mu \phi$$

$$\Rightarrow J^\mu{}_\rho = (-\partial^\mu \phi)(\partial_\rho \phi) - \delta^{\mu\nu}\eta_{\nu\rho}\mathcal{L}$$

$$J^{\mu\nu} = \eta^{\nu\rho} J^\mu{}_\rho = -\partial^\mu \phi \partial^\nu \phi - \delta^{\mu\nu}\mathcal{L} \equiv T^{\mu\nu}$$; energy-momentum tensor

$$\boxed{\begin{aligned} \partial_\mu J^\mu{}_\rho &= -\Box\phi(\partial_\rho \phi) - \partial^\mu \phi(\partial_\mu \partial_\rho \phi) - \partial_\rho \mathcal{L} \\ &= m^2 \phi(\partial_\rho \phi) - \partial^\mu \phi(\partial_\mu \partial_\rho \phi) \\ &\quad + (\partial_\rho \partial_\nu \phi)(\partial^\nu \phi) - m^2(\phi \partial_\rho \phi) = 0 \end{aligned}}$$

- Lorentz symmetry
  $\Rightarrow$ Preserve Minkowski metric

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \qquad \Lambda^\mu{}_\rho \simeq \delta^\mu_\rho + \omega^\mu{}_\rho$$

$$\Rightarrow (\delta^\mu_\rho + \omega^\mu{}_\rho)(\delta^\nu_\sigma + \omega^\nu{}_\sigma)\eta_{\mu\nu} = \eta_{\rho\sigma}$$

$$\Rightarrow \delta^\mu_\rho \, \omega^\nu{}_\sigma \, \eta_{\mu\nu} + \delta^\nu_\sigma \, \omega^\mu{}_\rho \, \eta_{\mu\nu} = 0$$

$\omega_{\rho\sigma} + \omega_{\sigma\rho} = 0 \Rightarrow \omega_{\mu\nu}$: anti-symmetric

6 independent elements

$$\phi'(x') = \phi(\Lambda^\mu{}_\nu x^\nu)$$

$$\delta_\omega \phi = \omega^\mu{}_\nu x^\nu \partial_\nu \phi(x)$$

$$\delta_\omega \mathcal{L} = \partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L})$$

$$\Rightarrow j^\mu = -\eta^{\mu\nu}\partial_\nu \phi(\omega^\rho{}_\sigma x^\sigma) - (\omega^\mu{}_\nu x^\nu \mathcal{L})$$

$$= \omega_{\rho\sigma}(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\nu)$$

$$= \omega_{\rho\sigma}\mathcal{M}^{\mu[\rho\sigma]}$$

$\mathcal{M}^{0[\rho\sigma]} \equiv M^{\rho\sigma} = T^{0\rho}X^\sigma - T^{0\sigma}X^\nu$ ⇒ angular momentum

-Global U(1)

$\phi \to e^{i\alpha}\phi$
$\phi^* \to e^{-i\alpha}\phi$ ⇒ $\delta_\alpha \phi = i\alpha$
$\delta_\alpha \phi^* = -i\alpha$ α : constant

$\mathcal{L} = -(\partial_\mu \phi)(\partial^\mu \phi^*) + m^2 \phi \phi^* \Rightarrow \delta\mathcal{L} = 0$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi^* \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\partial^\mu \phi$$

$\Rightarrow J^\mu = \underset{\substack{\wr \\ \text{"electric charge"}}}{\alpha}(-\phi\partial^\mu\phi^* + \phi^*\partial^\mu\phi)$ ⇝ probability current

Question : What if α become a function on the spacetime? : $\alpha = \alpha(x)$

⇒ $\mathcal{L}$ is not invariant.

⇒ Cure : Introduce a new field : Gauge field A_μ

IV. Local Gauge Theory and Classical Electrodynamics

IV.1. Recall: Electrodynamics

- Maxwell equati—

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases} \qquad \begin{cases} \vec{\nabla} \times \vec{B} - \frac{\partial E}{\partial t} = 4\pi \vec{J} \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

$(\vec{E}, \vec{B})$: Electric / Magnetic field

$(\rho, \vec{J})$: Electric density / Current

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial \phi}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{B}$$

Q. How can we describe the Maxwell equation in a covariant way?

$\Rightarrow$ Introduce four-potential $A_\mu = (-\phi, \vec{A})$

anti-symmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

four-current $J^\mu = (\rho, \vec{J})$

$$\Rightarrow \text{Maxwell eqn.} \Rightarrow \begin{cases} \partial_\mu F^{\mu\nu} = 4\pi \vec{J} \quad (\text{field eqn}) \\ \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \quad (\text{Bianchi idt.}) \end{cases}$$

IV.2. Local Gauge Theory in QM

Global U(1) gauge $\quad \psi \to e^{i\alpha}\psi$

$$-\frac{1}{2m}\vec{\nabla}^2\left(e^{i\alpha}\psi\right) = -\frac{1}{2m}\vec{\nabla}^2\psi$$

$\Downarrow$

Local U(1) gauge $\quad \psi \to e^{i\alpha(\vec{x})}\psi$

$$-\frac{1}{2m}\vec{\nabla}^2\left(e^{i\alpha(\vec{x})}\psi\right) \neq -\frac{1}{2m}\vec{\nabla}^2\psi$$

Q: How can we cure this problem?

A: Introduce new local function $\vec{A}$ which transforms under local U(1) gauge as $\begin{bmatrix} \vec{A} \to \vec{A} + i\vec{\nabla}\alpha(x) \\ \phi \to \phi - \partial_t \alpha(x) \end{bmatrix}$ $\quad \psi \to \psi e^{ie\alpha(x)}$

<u>Define</u> kinematic momentum

$$\vec{\pi} = \underset{\uparrow \text{ canonical momentum}}{\vec{p}} - e\vec{A} = -i\Big(\underbrace{\vec{\nabla} - ie\vec{A}}_{\vec{D}\ :\ \text{covariant derivative}}\Big)$$

$$H = \frac{1}{2m}\vec{\pi}^2 + e\phi$$

$$= \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\phi = -\frac{1}{2m}\left(\vec{\nabla}^2 - ie(\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}) - e^2\vec{A}^2\right) + e\phi$$

$$H \to -\frac{1}{2m}\Big(\vec{\nabla}^2 - ie\left(\vec{\nabla}\cdot\vec{A} + i\vec{\nabla}^2\alpha + \vec{A}\cdot\vec{\nabla} + (\vec{\nabla}\alpha)\cdot\vec{\nabla}\right)$$

$$- e^2\left(\vec{A}^2 + i\left(\vec{A}\cdot(\vec{\nabla}\alpha) + (\vec{\nabla}\alpha)\cdot\vec{A}\right) - (\vec{\nabla}\alpha)^2\right) + e\left(\phi - \frac{\partial\alpha}{\partial t}\right)$$

$$\vec{\nabla}^2(e^{ie\alpha}\psi) = \vec{\nabla}\cdot\left(ie(\vec{\nabla}\alpha)\psi + \vec{\nabla}\psi\right)e^{ie\alpha}$$

$$H\psi \longrightarrow -\frac{1}{2m}e^{ie\alpha}\Big[\Big(ie(\vec{\nabla}^2\alpha)\psi + ie(\vec{\nabla}\alpha)\cdot(\vec{\nabla}\psi) - \cancel{e^2(\vec{\nabla}\alpha)^2\psi}$$

$$+(\vec{\nabla}^2\psi) + ie\cancel{(\vec{\nabla}\psi)\cdot(\vec{\nabla}\alpha)}\Big)$$

$$-ie\Big(\vec{\nabla}\cdot\vec{A} + i\vec{\nabla}^2\alpha + \big(\cancel{ie\vec{A}\cdot(\vec{\nabla}\alpha)\psi} + \vec{A}\cdot(\vec{\nabla}\psi)\big)$$

$$+\cancel{ie(\vec{\nabla}\alpha)^2}\psi + \cancel{(\vec{\nabla}\alpha)\cdot(\vec{\nabla}\psi)}\Big)$$

$$-e^2\Big(\vec{A}^2\psi + i\big(\cancel{\vec{A}\cdot(\vec{\nabla}\alpha)} + (\vec{\nabla}\alpha)\vec{A}\big)\psi - (\vec{\nabla}\alpha)^2\psi\Big)\Big)$$

$$+e\left(\phi - \frac{\partial\alpha}{\partial t}\right)\psi\, e^{ie\alpha}$$

$\Rightarrow$ gauge invariant equation

$H\psi = -i\frac{\partial}{\partial t}\psi$ is invariant under local $U(1)$

<u>Cost</u> : Replace $\vec{p} \longrightarrow \vec{\pi}$

Introduce new local field $\underbrace{\vec{A},\ \phi}_{\text{gauge field}}$

IV.3. Lagrangian Description of Maxwell Theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 4\pi A_\mu j^\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Field equation $\quad \partial_\mu F^{\mu\nu} = 4\pi j^\nu \rightsquigarrow \partial_\nu j^\nu = 0$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 4\pi j^\nu$$

<u>Note</u>. We can find electrodynamic Lagrangian

$$\mathcal{L} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

- Gauge invariance

$$A^\mu \to A^\mu + \partial^\nu\alpha(x) \qquad \rightsquigarrow F_{\mu\nu} \to F_{\mu\nu}$$

Choose Lorenz gauge condition

$$\partial_\mu A^\mu = 0 \longrightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu(\partial^\mu\alpha) = 0$$

$$\Rightarrow \Box\alpha = 0$$

$\Rightarrow \Box A^\mu = 4\pi j^\mu$: Klein-Gordon equation with current j^μ

IV. 4. Scalar Electrodynamics

Local U(1) gauge invariant theory

Covariant derivative

$$\mathcal{L}_{scalar} = -\frac{1}{2}(D_\mu\phi)(D^\mu\phi)^* + \frac{1}{2}m^2|\phi|^2 \qquad D_\mu\phi = (\partial_\mu - iqA_\mu)\phi$$

$$D_\mu\phi^* = (\partial_\mu + iqA_\mu)\phi$$

$\mathcal{L}_{scalar}$ is invariant under $\phi \to e^{iq\alpha(x)}\phi$, $\phi \to e^{-iq\alpha(x)}$

$A_\mu \to A_\mu + \partial_\mu\alpha(x)$

$\Rightarrow$ Noether theorem

$$j^\mu = iq(\phi^* D^\mu\phi - \phi(D^\mu\phi)^*)$$

If we want to consider A_μ as a dynamical field, we should add $\mathcal{L}_{EM}$ to $\mathcal{L}_{scalar}$

$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{EM}$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)(D^\mu\phi)^* + \frac{1}{2}m^2|\phi|^2$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}m^2|\phi|^2$$

$$+ iqA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) - q^2A^2|\phi|^2$$

V. Spontaneous Symmetry Breaking

V.1. Prelude : Ising model and Landau theory

(Ising, 1924) Description of (ferro →) magnetism by spin operator

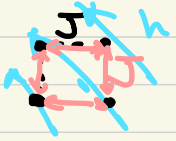
$$H(s_i ; \{K\}) = - \sum_{\{n\}} K^{(n)} \sigma^{(n)}_{i(n)}$$

σ : local operator
K : coupling constant

ex) $n=1$ $\quad \sigma^{(1)}_i = s_i$
$n=2$ $\quad \sigma^{(2)}_{ij} = s_i s_j$
$n=3$ $\quad \sigma^{(3)}_{ijk} = s_i s_j s_k \quad \ldots$

We only consider $n=1$ and $n=2$ (up to nearest neighbour)

$$H(s; J, h, T) = -h \sum_{i=1}^{N} s_i - \sum_{i=1}^{N-1} J_{ij} s_i s_{i+1}$$



Zeeman $\qquad$ exchange interaction

* h breaks time-reversal symmetry
($s \to -s$)

$J>0$ Ferromagnetic Ising
$J<0$ Anti - 〃

Suppose $J=0$ (paramagetic case)

$$Z[s; J=0, h, T] = \prod_{i=1}^{N} (e^{h\beta} + e^{-h\beta}) = (2\cosh^N h\beta)$$

$$m = -\frac{1}{N} \frac{\partial F}{\partial h} = \tanh h\beta$$

magnetism $\langle s_i \rangle$

When $J \neq 0$, each spin feels not only the external field h, but also an effective interaction heff due to the all other spins.
But, in general, heff is unknown.

$$H = -\sum J_{ij} s_i s_j - h \sum s_i$$

$$= -\sum_i s_i \underbrace{\left(h + \sum_j J_{ij} s_j \right)} \equiv -\sum_i h_i^{eff} s_i$$

$$h_i^{eff} = h + \sum_j J_{ij} s_j = h + \sum_j J_{ij} \langle s_j \rangle + \underbrace{\cancel{\sum_j J_{ij} (s_j - \langle s_j \rangle)}}$$

$$= h + 2dJm$$

⇓
Assume
No fluctuation on s_j ⇒ MFT

order parameter
$\hat{\phi}$

$$\therefore \; m = \tanh(\beta h_{eff}) = \tanh \beta (h + 2dJm)$$

- Landau theory
  - Assumptions

$h, J, t = \frac{T - T_c}{T_c}$

Let f_L : energy functional with respect to the $\{K\}$ and order parameter ϕ → m

i) f_L obeys symmetries of the system.

ii) Near critical temperature T_c, f_L is an analytic function for both $\{K\}$ and ϕ

$$f_L = \sum a_n(\{K\})\phi^n$$

- Landau theory for Ising model

If $h=0$ ⇒ time reversal symmetry exists

⇒ we demand $f_L[\phi] = f_L[-\phi]$

$$\Rightarrow f_L = a_0 + a_2\phi^2 + a_4\phi^4 + \cdots$$

- We set $a_0 = 0$ since it is just an energy shift.
- a_4 must be positive.

$$\Rightarrow f_L = a\phi^2 + b\phi^4$$

At the minimum : $\left.\frac{\partial f_L}{\partial \phi}\right|_{\phi=\phi_*} = 0 \Rightarrow \phi_* = 0$ or $\phi_* = \pm\sqrt{\frac{-a}{2b}}$

If $a(t) \sim a_0 + a_1 t + \mathcal{O}(t^2)$

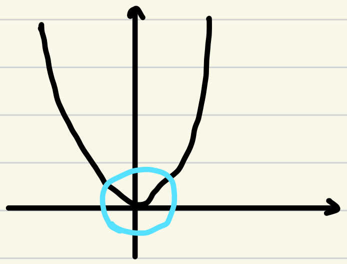
Since $\phi(t=0) = 0 \Rightarrow a_0 = 0$

$$\Rightarrow f_L = at\phi^2 + b\phi^4$$

① $t>0$ ⇒ Global minimum occurs at $\phi_* = 0$

$\Updownarrow$

$T > T_c$

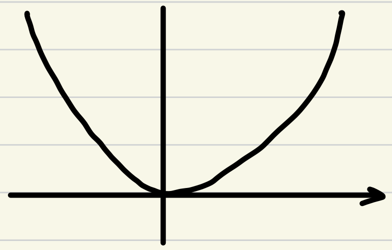


$f_L \sim \phi^2$

② $t = 0$

$\Updownarrow$

$T = T_c$

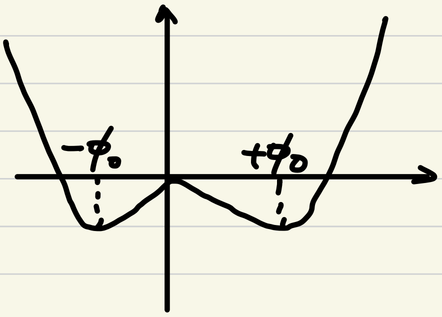


$f_L \sim \phi^4$

③ $t<0$

$\Updownarrow$

$T < T_c$



$\phi_* = 0$ or $\pm\sqrt{\frac{-at}{2b}} \equiv \pm\phi_0$

- $\phi = 0$ is not only no longer the global minimum, but also the local maximum.
- There exist two minimum related to the symmetry $\phi \to -\phi$

⇒ Spontaneous symmetry breaking

Ⅴ.2. Spontaneous Symmetry Breaking

- Double-well potential

• Self-interacting field

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + \underbrace{\frac{1}{2}m^2\phi^2} + \underbrace{\overset{\nearrow\ \text{self-interacting term}}{\widetilde{V(\phi)}}}_{\frac{g}{3}\phi^3+\frac{\lambda}{4}\phi^4+\cdots}$$

__Note__ Coefficient of ϕ^2 term is called mass of field.

$\Rightarrow$ Equation of motion: $(\Box - m^2)\phi = V'(\phi)$

• Maxican hat potential

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \overbrace{\underset{\sim\sim\sim\sim\sim}{\left(-\frac{a^2}{2}\phi^2 + \frac{b^2}{4}\phi^4\right)}}^{V(\phi)}$$

· Symmetry

* Translation] Poincaré
* Lorentz
* $\varphi \to -\varphi$ — $\mathbb{Z}_2$

$V'(\phi) = 0 \Rightarrow \phi = 0$ or $\pm\frac{a}{b}$

$\phi = 0$ ↓ Both Poincaré and $\mathbb{Z}_2$ invariant

$\pm\frac{a}{b}$ ↘ Poincaré invariant, $\mathbb{Z}_2$ - not invariant

- Spontaneous

Noether current: Energy functional

$$E = \int_V dV\left(\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{a^2}{2}\phi^2 + \frac{b^2}{4}\phi^4\right)$$

$$\delta E = \int_V dV\{(\partial_0\phi)(\partial_0\delta\phi) + (\vec{\nabla}\phi)(\vec{\nabla}\delta\phi) - a^2\phi\,\delta\phi + b^2\phi^3\delta\phi)$$

$$= \int_V dV\,\delta\phi\left[-(\partial_0\phi)^2 - (\vec{\nabla}\phi)^2 - a^2\phi + b^2\phi^3\right]$$

$\Rightarrow \phi = 0$ or $\pm\frac{a}{b}$: critical point

$\phi = 0$ ↓
· local maximum
· unstable point

$\pm\frac{a}{b}$ ↓
· global minimum
· stable point
$\Rightarrow$ ground state of classical field

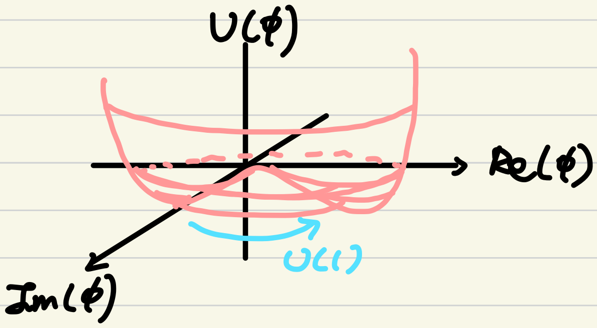
$\Rightarrow$ In this case, ground state is less symmetric $\text{Poin}\cancel{\times \mathbb{Z}_2} \to \text{Poin}$

* If $a^2 \to -a^2$, there is no SSB.

- SSB for Complex field

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)^* - \overbrace{\Big(\underbrace{-\frac{a^2}{2}|\phi|^2}_{\text{assume } a,b>0} + \frac{b^2}{4}|\phi|^4\Big)}^{V(\phi)}$$

· Symmetry : Poin × U(1) $\qquad \phi \to e^{i\alpha}\phi$



$$V'(\phi) = 0 \Rightarrow \frac{b^2}{4}\varphi|\varphi|^2 - \frac{1}{2}a^2\varphi = 0$$

$$\Rightarrow \begin{cases} |\varphi| = \frac{a}{b} & \Rightarrow \varphi = \frac{a}{b}e^{i\theta} \rightsquigarrow \text{topologically circle} \\ \varphi = 0 & \rightsquigarrow \varphi \sim \varphi e^{i\alpha+\theta} \end{cases}$$

· Energy functional

$$E = \int d^3\vec{x}\left(\frac{1}{2}|\partial_0\phi|^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 - \frac{a^2}{2}|\phi|^2 + \frac{b^2}{4}|\phi|^4\right)$$

Let $\phi = \rho(x)e^{i\omega(x)}$

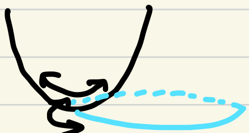
$$E = \int d^3\vec{x}\Big[\frac{1}{2}(\partial_0\rho)^2 + \frac{1}{2}(\vec{\nabla}\rho)^2 + \frac{\rho^2}{2}\big((\partial_0\omega)^2 + (\vec{\nabla}\omega)^2\big) - \frac{a^2}{2}\rho^2 + \frac{b^2}{4}\rho^4\Big]$$

Local variation

$$\delta E = -\frac{1}{4}\frac{a^4}{b^2} + \int d^3\vec{x}\Big[\frac{1}{2}(\delta\partial_0\rho)^2 + \frac{1}{2}(\delta\vec{\nabla}\rho)^2 + \frac{1}{2}\left(\frac{a}{b}\right)^2\big((\delta\partial_0\omega)^2 + (\delta\vec{\nabla}\omega)^2\big) + a^2(\delta\rho)^2\Big]$$

$\Rightarrow \delta\rho = 0,\ \delta\omega = \text{const} \Rightarrow$ No energy change

Otherwise, energy is increased.



V.3. Goldstone Theorem

- Linearisation

Extremum : ϕ_0

Near extremum point (stable) $\phi(x) = \phi_0(x) + \varphi(x)$

↳ satisfying eom

Ex) · Gravity theory $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$

↳ Graviton in flat space

$$G_{\mu\nu}[g] = \underbrace{G_{\mu\nu}[\eta]}_{\overset{\shortparallel}{0}} + \underbrace{G_{\mu\nu}[\eta, h]}_{\overset{\shortparallel}{0}}$$

· ϕ^4 - theory of real scalar

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

EoM $\Box\phi + m^2\phi + \lambda\phi^3 = 0$ Non-linear PDE

$$\phi = \phi_0 + \varphi$$

$$\Box\phi_0 + \Box\varphi + m^2\phi_0 + m^2\varphi + \lambda(\phi_0 + \varphi)^3 = 0$$

$$\Rightarrow \Box\varphi + m^2\varphi + \lambda(\underbrace{\cancel{3\phi_0\varphi^2} + 3\phi_0^2\varphi + \cancel{\varphi^3}}_{\text{assuming small approximation}}) = 0$$

$$\Rightarrow \Box\varphi + (\underbrace{m^2 + 3\lambda\phi_0^2}) \varphi = 0$$ Linear PDE

↳ φ is a scalar field with eff. mass $m^2 + 3\lambda\phi_0^2 \equiv m_{eff}$

In formal case $m^2 = -a^2$, $\lambda \to b^2$

$\varphi_0 = \pm a/b \Rightarrow m_{eff} = 2a^2 > 0 \Rightarrow$ stable

$\varphi_0 = 0 \Rightarrow m_{eff} = -a^2 < 0 \Rightarrow$ unstable

- Goldstone Theorem

Complex scalar theory $\phi = \rho(x) e^{i\omega(x)}$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{2}\rho^2\partial_\mu\omega\,\partial^\mu\omega - \frac{1}{2}a^2\rho^2 + \frac{1}{4}b^2\rho^4$$

$$\Rightarrow \begin{cases} \delta_\rho\mathcal{L} = \Box\rho - \rho\,\partial_\mu\omega\,\partial^\mu\omega - a^2\rho + b^2\rho^3 = 0 \\ \delta_\omega\mathcal{L} = \partial_\mu(\rho^2\partial^\mu\omega) = 0 \quad \Rightarrow \text{conservation law} \end{cases}$$

$\rho_0 = \frac{a}{b}$, $\omega_0 = \text{const}$

$\Rightarrow \rho = \frac{a}{b} + \varphi \qquad \omega = \omega_0 + \theta$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\left(\frac{a}{b}\right)^2(\partial_\mu\theta)(\partial^\mu\theta) - \frac{1}{2}\frac{a^4}{b^2} + a^2\varphi^2$$

$+ \text{(Higher order)}$

eom: $\left[\begin{array}{ll} \Box \rho - 2a^2\rho = 0 & \Rightarrow \text{massive field} \\ \Box \theta = 0 & \Rightarrow \text{massless field} \end{array}\right.$

Goldstone theorem : Broken symmetry ⇒ Massless field
⇒ Goldstone field

V.4. Abelian Higgs Model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)(D^\mu\phi)^* - V(\phi)$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\phi)(\partial^\mu\phi)^* - iqA_\mu(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$$

$$- \quad q^2A^2|\phi|^2 - \frac{a^2}{2}|\phi|^2 - \frac{b^2}{4}|\phi|^4$$

$$\phi = \rho(x)e^{i\omega(x)}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\rho\,\partial^\mu\rho - \rho^2\partial_\mu\omega\,\partial^\mu\omega$$

$$-iqA_\mu\Big(\rho(x)\cancel{e^{i\omega}}\big[\cancel{\partial^\mu\rho\, e^{-i\omega}} + \rho(-i\partial^\mu\omega)\cancel{e^{-i\omega}}\big]$$

$$-\rho(x)\cancel{e^{-i\omega}}\big[\cancel{\partial^\mu\rho}\,\cancel{e^{+i\omega}} + \rho(+i\partial^\mu\omega)\cancel{e^{+i\omega}}\big]\Big)$$

$$-q^2A^2\rho^2 - \frac{a^2}{2}\rho^2 - \frac{b^2}{4}\rho^4$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\rho\,\partial^\mu\rho - \rho^2\partial_\mu\omega\,\partial^\mu\omega$$

$$-2qA_\mu\rho^2\partial^\mu\omega - q^2A^2\rho^2 - \frac{a^2}{2}\rho^2 - \frac{b^2}{4}\rho^4$$

Ground state $\rho(x) = \frac{a}{b}$, $\omega(x) = \omega_0$, $A_\mu = 0$
(Poincaré invariant)

Linearisation: $\rho(x) = \frac{a}{b} + H(x)$, $\omega(x) = \omega_0 + \pi(x)$ $A_\mu = 0 + a_\mu(x)$

Introduce a new field $B_\mu = a_\mu + \frac{1}{q}\partial_\mu\pi$

$$\mathcal{L} \approx \underbrace{-\frac{1}{4}F^B_{\mu\nu}F^{B\,\mu\nu} - \left(\frac{aq}{b}\right)^2 B_\mu B^\mu}_{\text{Spin-1 Lagrangian with mass } \frac{aq}{b}\equiv\mu} \underbrace{-\frac{a^2}{2}H^2 - \partial_\mu H\,\partial^\mu H}_{\Downarrow}$$

Scalar field lagrangian

⇒ Proca Lagrangian

→ eom for B_μ $\left[\begin{array}{l} (\Box - k^2)B_\mu = 0 \\ \partial^\mu B_\mu = 0 \end{array}\right] \Rightarrow$ 3 independent component (scalar dof)

- Massive vector ⇒ No gauge invariance
- Mass of B is related to charge q and mass of H.

a=1,2,3,0 $B_\mu^a \left[\begin{matrix} A \\ W^{\pm} \\ Z \end{matrix}\right.$

⇒ Near ground state : Massless vector ⇒ Massive vector : B_μ

~~U(1)~~ Scalar $\underbrace{\text{Massive scalar}}_{\text{Higgs Phenomena}}$: H

Higgs

SU(2) × U(1) : Electroweak